

DECORRELATION ESTIMATES FOR RANDOM SCHRÖDINGER OPERATORS WITH NON RANK ONE PERTURBATIONS

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ABSTRACT. We prove decorrelation estimates for generalized lattice Anderson models on \mathbb{Z}^d constructed with finite-rank perturbations in the spirit of Klopp [7]. These are applied to prove that the local eigenvalue statistics ξ_E^ω and $\xi_{E'}^\omega$, associated with two energies E and E' satisfying $|E - E'| > 4d$, are independent. That is, if I, J are two bounded intervals, the random variables $\xi_E^\omega(I)$ and $\xi_{E'}^\omega(J)$, are independent and distributed according to a compound Poisson distribution whose Lévy measure has finite support. We also prove that the extended Minami estimate implies that the eigenvalues in the localization region have multiplicity at most the rank of the perturbation.

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1. STATEMENT OF THE PROBLEM AND RESULTS

We consider random Schrödinger operators $H^\omega = \mathcal{L} + V_\omega$ on the lattice Hilbert space $\ell^2(\mathbb{Z}^d)$ (or, for matrix-valued potentials, on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{m_k}$), and prove that certain natural random variables associated with the local eigenvalue statistics around two distinct energies E and E' , in the region of complete localization Σ_{CL} and with $|E - E'| > 4d$, are independent. From previous work [8], these random variables distributed according to a compound Poisson distribution. The operator \mathcal{L} is the discrete Laplacian on \mathbb{Z}^d , although this can be generalized. For these lattice models, the random potential V_ω has the form

$$(V_\omega f)(j) = \sum_{i \in \mathcal{J}} \omega_i (P_i f)(j), \quad (1.1)$$

where $\{P_i\}_{i \in \mathcal{J}}$ is a family of finite-rank projections with the same rank $m_k \geq 1$, the set \mathcal{J} is a sublattice of \mathbb{Z}^d , and $\sum_{i \in \mathcal{J}} P_i = I$. We assume that $P_i = U_i P_0 U_i^{-1}$, for $i \in \mathcal{J}$, where U_i is the unitary implementation of the translation group $(U_i f)(k) = f(k + i)$, for $i, k \in \mathbb{Z}^d$. The coefficients $\{\omega_i\}$ are a family of independent, identically distributed (iid) random variables with a bounded density of compact support on a product probability space Ω with probability measure \mathbb{P} . It follows from the conditions above that the family of random Schrödinger operators H^ω is ergodic with respect to the translations generated by \mathcal{J} .

One example on the lattice is the polymer model. For this model, the projector $P_i = \chi_{\Lambda_k(i)}$ is the characteristic function on the cube $\Lambda_k(i)$ of side length k centered at $i \in \mathbb{Z}^d$. The rank of P_i is $(k + 1)^d$ and the set \mathcal{J} is chosen so that $\cup_{i \in \mathcal{J}} \Lambda_k(i) = \mathbb{Z}^d$. Another example is a matrix-valued model for which P_i , $i \in \mathbb{Z}^d$, projects onto an m_k -dimensional subspace, and $\mathcal{J} = \mathbb{Z}^d$. The corresponding Schrödinger operator is

$$H^\omega = \mathcal{L} + \sum_{i \in \mathcal{J}} \omega_i P_i, \quad (1.2)$$

where \mathcal{L} is the discrete lattice Laplacian Δ on $\ell^2(\mathbb{Z}^d)$, or $\Delta \otimes I$ on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{m_k}$ (or, more generally, $\Delta \otimes A$, where A is a nonsingular $m_k \times m_k$ matrix), respectively. In the following, we denote by $H_{\omega, \ell}$ (or simply as H_ℓ omitting the ω) the matrices $\chi_{\Lambda_\ell} H^\omega \chi_{\Lambda_\ell}$ and similarly $H_{\omega, L}, H_L$ by replacing ℓ with L , for positive integers ℓ and L .

A lot is known about the eigenvalue statistics for random Schrödinger operators on $\ell^2(\mathbb{R}^d)$. When the projectors P_i are rank one projectors, the local eigenvalue statistics in the localization regime has been proved to be given by a Poisson process by Minami [9] (see also Molchanov [10] for a model on \mathbb{R} and Germinet-Klopp [5] for a comprehensive discussion and additional results). For the non rank one case, Tautenhahn and Veselić [13] proved a Minami estimate for certain models that may be described as weak perturbations of the rank one case. The general non finite rank case was studied by the authors in [8] who proved that, roughly speaking, the local eigenvalue statistics are compound Poisson. This result also holds for random Schrödinger operators on \mathbb{R}^d .

In this paper, we further refine these results for lattice models with non rank one projections and prove, roughly speaking, that the processes associated with two distinct energies are independent. Klopp [7] proved decorrelation estimates for lattice models in any dimension. He applied them to show that the local eigenvalue point processes at distinct energies converge to independent Poisson processes (in dimensions $d > 1$ the energies need to be far apart as is the case for the models studied here). Shirley [12] extended the family of one-dimensional lattice models for which the decorrelation estimate may be proved to include alloy-type models with correlated random variables, hopping models, and certain one-dimensional quantum graphs.

1.1. Asymptotic independence and decorrelation estimates. The main result is the asymptotic independence of random variables associated with the local eigenvalue statistics centered at two distinct energies E and E' satisfying $|E - E'| > 4d$.

We note that in one-dimension there are stronger results and the condition $|E - E'| > 4d$ is not needed. Our results are inspired by the work of Klopp [7] for the Anderson models on \mathbb{Z}^d and of Shirley [12] for related models on \mathbb{Z}^d . The condition $|E - E'| > 4d$ requires that the two energies be fairly far apart. For example, if $\omega_0 \in [-K, K]$ so that the deterministic spectrum $\Sigma = [-2d - K, 2d + K]$, the region of complete localization Σ_{CL} is near the band edges $\pm(2d + K)$. In this case, one can consider E and E' near each of the band edges. Our main result on asymptotic independence is the following theorem.

Theorem 1.1. *Let $E, E' \in \Sigma_{\text{CL}}$ be two distinct energies with $|E - E'| > 4d$. Let $\xi_{\omega, E}$, respectively, $\xi_{\omega, E'}$, be a limit point of the local eigenvalue statistics centered at E , respectively, at E' . Then these two processes are independent. For any bounded intervals $I, J \in \mathcal{B}(\mathbb{R})$, the random variables $\xi_{\omega, E}(I)$ and $\xi_{\omega, E'}(J)$ are independent random variables distributed according to a compound Poisson process.*

We refer to [5] for a description of the region of complete localization Σ_{CL} . For information on Lévy processes, we refer to the books by Applebaum [1] and by Bertoin [2]. Theorem 1.1 follows (see section 4) from the following decorrelation estimate. We assume that $L > 0$ is a positive integer, and that $\ell := [L^\alpha]$ is the greatest integer less than L^α for an exponent $0 < \alpha < 1$. For polymer type models, we assume that m_k divides L and ℓ .

Proposition 1.1. *We choose positive numbers (α, β) satisfying (3.32) and length scales L and $\ell := [L^\alpha]$ as described above. For a pair of energies $E, E' \in \Sigma_{\text{CL}}$, the region of complete localization, with $|E - E'| > 4d$, and bounded intervals $I, J \subset \mathbb{R}$, we define $I_L(E) := L^{-d}I + E$ and $J_L(E') := L^{-d}J + E'$ as two scaled energy intervals centered at E and E' , respectively. We then have*

$$\mathbb{P}\{(\text{Tr}E_{H_{\omega, \ell}}(I_L(E)) \geq 1) \cap (\text{Tr}E_{H_{\omega, \ell}}(J_L(E')) \geq 1)\} \leq \frac{C_0}{L^{2d(1-2\beta-\alpha)}}. \quad (1.3)$$

The extended Minami estimate [8] implies that we need to estimate:

$$\mathbb{P}\{(\text{Tr}E_{H_{\omega, \ell}}(I_L(E)) \leq m_k) \cap (\text{Tr}E_{H_{\omega, \ell}}(J_L(E')) \leq m_k)\} \quad (1.4)$$

In fact, we consider the more general estimate:

$$\mathbb{P}\{(\text{Tr} E_{H_{\omega,L}}(I_L(E)) = k_1) \cap (\text{Tr} E_{H_{\omega,L}}(J_L(E')) = k_2)\}, \quad (1.5)$$

where k_1, k_2 are positive integers independent of L .

We allow that there may be several eigenvalues in $I_L(E)$ and $J_L(E')$ with nontrivial multiplicities. To deal with this, we introduce the mean trace of the eigenvalues $E_j(\omega)$ of $H_{\omega,\ell}$ in the interval $I_L(E)$:

$$\mathcal{T}(\omega) := \frac{\text{Tr}(H_{\omega,\ell} E_{H_{\omega,\ell}}(I_L(E)))}{\text{Tr}(E_{H_{\omega,\ell}}(I_L(E)))} = \frac{1}{k_1} \sum_{j=1}^{k_1} E_j(\omega), \quad (1.6)$$

where $k_1 := \text{Tr}(E_{H_{\omega,\ell}}(I_L(E)))$ is the number of eigenvalues, including multiplicity, of $H_{\omega,\ell}$ in $I_L(E)$. Similarly, we define

$$\mathcal{T}'(\omega) := \frac{\text{Tr}(H_{\omega,\ell} E_{H_{\omega,\ell}}(J_L(E')))}{\text{Tr}(E_{H_{\omega,\ell}}(J_L(E')))} = \frac{1}{k_2} \sum_{j=1}^{k_2} E_j(\omega). \quad (1.7)$$

We will show in section 2 that these weighted sums behave like an effective eigenvalues in each scaled interval.

As another application of the extended Minami estimate, we prove that the multiplicity of eigenvalues in Σ_{CL} is at most the multiplicity of the perturbations m_k in dimensions $d \geq 1$. The proof of this fact follows the argument of Klein and Molchanov [6]. For $d = 1$, Shirley [12] proved that the usual Minami estimate holds for the dimer model so the eigenvalues are almost surely simple.

1.2. Contents. We present properties of the weighted average of eigenvalues in section 2, including gradients and Hessian estimates. The proof of the main technical result, Proposition 1.1, is presented in section 3. The proof of asymptotic independence is given in section 4. We show in section 5 that the argument of Klein-Molchanov [6] applies to higher rank perturbations and implies that the multiplicity of eigenvalues in Σ_{CL} is at most m_k , the uniform rank of the perturbations.

2. ESTIMATES ON WEIGHTED SUMS OF EIGENVALUES

In this section, we present some technical results on weighted sums of eigenvalues of $H_{\omega,\ell}$ defined in (1.6)-(1.7). These are used in section 4 to prove the main technical result (1.3).

2.1. Properties of the weighted trace. When the total number of eigenvalues of $H_{\omega,\ell}$ in $J_L(E) := L^{-d}I + E$ is k_1 , we get,

$$\mathcal{T}(\omega) := \mathcal{T}_\ell(E, k_1) := \mathcal{T}_\ell(E, k_1, \omega) = \frac{1}{k_1} \sum_{j=1}^{k_1} E_j(\omega), \quad (2.8)$$

for eigenvalues $E_j(\omega) \in J_L(E)$. Properties (1)-(3) below are valid for the similar expression obtained by replacing k_1 with k_2 , the interval I with J , and the energy E with E' . We will write

$$\mathcal{T}'(\omega) := \mathcal{T}_\ell(E', k_2) := \mathcal{T}(E', k_2, \omega).$$

The weighted eigenvalue average behaves like an effective eigenvalue in the following sense:

- (1) $\mathcal{T}_\ell(E, k_1, \omega) \in J_L(E)$, so the weighted average of the eigenvalue cluster in $J_L(E)$ behaves as an eigenvalue in $J_L(E)$.
- (2) Let $E_j(\omega) \in J_L(E)$ be an eigenvalue of multiplicity m_j . Then a derivative may be computed as follows. Let $\varphi_{j,i}$, for $i = 1, \dots, m_j$, be an orthonormal basis of the eigenspace for $E_j(\omega)$. Then,

$$0 = \frac{\partial}{\partial \omega_s} \sum_{i=1}^{m_j} \langle \varphi_{j,i}, (H_{\omega,\ell} - E_j(\omega)) \varphi_{j,i} \rangle, \quad (2.9)$$

so we obtain

$$\frac{\partial E_j(\omega)}{\partial \omega_s} = \frac{1}{m_j} \sum_{i=1}^{m_j} \|P_s \varphi_{j,i}\|^2, \quad (2.10)$$

where P_s is the projector associated with the random variable ω_s .

- (3) Suppose there are \hat{k}_1 distinct eigenvalues in $J_L(E)$ each with multiplicity m_j so $\sum_{j=1}^{\hat{k}_1} m_j = k_1$. Then, we have

$$\frac{\partial \mathcal{T}_\ell(E, k_1, \omega)}{\partial \omega_s} = \frac{1}{k_j} \sum_{j=1}^{\hat{k}_1} \sum_{i=1}^{m_j} \|P_s \varphi_{j,i}\|^2 \geq 0. \quad (2.11)$$

This shows that $\mathcal{T}_\ell(E, k_1, \omega)$ is non-decreasing as a function of ω_s .

- (4) It follows from (2.11) that the ω -gradient of the weighted trace is normalized: $\|\nabla_\omega \mathcal{T}(\omega)\|_{\ell^1} = 1$.

Remark 1. It follows from property (1) above and the fact that the intervals $I_L(E)$ and $J_L(E)$ are $\mathcal{O}(L^{-d})$, that if $|E - E'| > 4d$, then $|\mathcal{T}(\omega) - \mathcal{T}(\omega')| > 4d - cL^{-d}$, for some $c > 0$. We will use this result below.

2.2. Variational formulae. We can estimate the variation of the mean trace with respect to the random variables as follows. The ω -directional derivative is

$$\begin{aligned} \omega \cdot \nabla_\omega (\mathcal{T}(\omega) - \mathcal{T}'(\omega)) &= \frac{1}{k_1} \sum_{i=1}^{k_1} \omega \cdot \nabla_\omega E_i(\omega) - \frac{1}{k_2} \sum_{j=1}^{k_2} \omega \cdot \nabla_\omega E_j(\omega) \\ &= \mathcal{T}(\omega) - \mathcal{T}'(\omega) - \frac{1}{k_1} \sum_{i=1}^{k_1} \langle \varphi_i, (-\Delta) \varphi_i \rangle \\ &\quad + \frac{1}{k_2} \sum_{j=1}^{k_2} \langle \varphi_j, (-\Delta) \varphi_j \rangle. \end{aligned} \quad (2.12)$$

On the lattice, the absolute value of each sum involving the Laplacian may be bounded above by $2d$. If we assume that

$$|\mathcal{T}(\omega) - \mathcal{T}'(\omega)| \geq \Delta E$$

then we obtain from (2.12),

$$\begin{aligned} \Delta E - 4d &\leq |\mathcal{T}(\omega) - \mathcal{T}'(\omega)| - 4d \\ &\leq |\omega \cdot \nabla_\omega (\mathcal{T}(\omega) - \mathcal{T}'(\omega))|. \end{aligned} \quad (2.13)$$

As the number of components of ω is bounded by ℓ^d and $|\omega_j| \leq K$, it follows by Cauchy-Schwartz inequality that

$$\|\nabla_\omega(\mathcal{T}(\omega) - \mathcal{T}'(\omega))\|_2 \geq \frac{\Delta E - 4d}{K} \frac{1}{(2\ell + 1)^{d/2}}. \quad (2.14)$$

We also obtain an ℓ^1 lower bound:

$$\|\nabla_\omega(\mathcal{T}(\omega) - \mathcal{T}'(\omega))\|_1 \geq \frac{\Delta E - 4d}{K}. \quad (2.15)$$

2.3. Hessian estimate. The Hessian of $\mathcal{T}(\omega)$ has ij^{th} matrix elements given by

$$\text{Hess}(\mathcal{T})_{ij} = \frac{1}{k_1} \sum_{m=1}^{k_1} \frac{\partial^2}{\partial \omega_i \partial \omega_j} E_m(\omega). \quad (2.16)$$

It is convenient to compute this using trace notation. Let P_E denote the spectral projection onto the eigenspace of $H_{\omega,\ell}$ corresponding to the eigenvalues $E_m(\omega)$ in $J_L(E)$. Let γ_E be a simple closed contour containing only these eigenvalues of $H_{\omega,\ell}$ with a counter-clockwise orientation. Since the weighted mean of the eigenvalues may be expressed as

$$\mathcal{T}(\omega) = \frac{1}{k_1} \text{Tr} H_{\omega,\ell} P_E,$$

and the projection has the representation

$$P_E = \frac{1}{2\pi i} \int_{\gamma_E} R(z) dz, \quad R(z) := (H_{\omega,\ell} - z)^{-1},$$

it follows that

$$\frac{\partial}{\partial \omega_j} \mathcal{T}(\omega) = \frac{-1}{2\pi i k_1} \int_{\gamma_E} \text{Tr}\{R(z) P_j R(z)\} z dz, \quad (2.17)$$

where P_j is the finite-rank projector associated with site j or block j , depending on the model. Computing the second derivative, the matrix elements of the Hessian of $\mathcal{T}(\omega)$ are

$$\begin{aligned} \text{Hess}(\mathcal{T})_{ij} &= \frac{1}{2\pi i k} \int_{\gamma_E} \text{Tr}\{R(z) P_i R(z) P_j R(z) \\ &\quad + R(z) P_j R(z) P_i R(z)\} z dz \end{aligned} \quad (2.18)$$

This formula will provide the equivalent of Lemma 2.3 [7], for both $\text{Hess}(\mathcal{T})_{ij}, \text{Hess}(\mathcal{T}')_{ij}$.

Lemma 2.1. *The Hessian of the weighted average $\mathcal{T}(\omega)$ of the eigenvalues of $H_\ell(\omega)$ in an interval of order L^{-d} satisfies the bound:*

$$\begin{aligned} \|\text{Hess}(\mathcal{T})_{ij}\|_{\ell^\infty \rightarrow \ell^1} &\leq |\gamma_E|^2 \sup_{z \in \gamma_E} \|P_i R(z)^2 P_j\|_1 \|P_j R(z) P_i\|_1 \\ &\leq C \frac{L^{-2d}}{(\text{dist}(\gamma_E, \sigma(H_{\omega,\ell})))^3}. \end{aligned} \quad (2.19)$$

Since the Wegner estimate insures that $\text{dist}(\gamma_E, \sigma(H_{\omega, \ell})) \sim \ell^{-d}$ with probability greater than $1 - C_W(\ell/L)^d$, we obtain

$$\|\text{Hess}(\mathcal{T})_{ij}\|_{\ell^\infty \rightarrow \ell^1} \leq CL^{-2d} \ell^{3d} \leq CL^{3d\alpha-2d}, \quad (2.20)$$

so if $0 < \alpha < 2/3$, the Hessian vanishes as $L \rightarrow \infty$. The above statements are also valid for $\mathcal{T}'(\omega)$.

3. PROOF OF PROPOSITION 1.1

In this section, we prove the technical result, Proposition 1.1. We let $X_\ell(I_L(E)) := \text{Tr} E_{H_{\omega, \ell}}(I_L(E))$, $X_\ell(J_L(E')) := \text{Tr} E_{H_{\omega, \ell}}(J_L(E'))$. Then, we show

$$\mathbb{P}\{(X_\ell(I_L(E)) \geq 1) \cap (X_\ell(J_L(E')) \geq 1)\} \leq C_0 \frac{1}{L^{2d(1-2\beta-\alpha)}}, \quad (3.21)$$

for positive numbers (α, β) satisfying (3.32).

3.1. Reduction via the extended Minami estimate. Let $\chi_A(\omega)$ be the characteristic function on the subset $A \subset \Omega$. In this section, we write $J_L(E) := L^{-d}J + E$ since we are dealing with one interval. We use an extended Minami estimate of the form

$$\mathbb{E}\{\chi_{\{\omega \mid X_\ell(J_L(E)) \geq m_k+1\}} X_\ell(J_L(E))(X_\ell(J_L(E)) - m_k) \geq 1\} \leq C_M \left(\frac{\ell}{L}\right)^{2d},$$

as follows from [8].

Lemma 3.1. *Under the condition that the projectors have uniform dimension $m_k \geq 1$, we have*

$$\mathbb{P}\{X_\ell(J_L(E)) > m_k\} \leq C_M \left(\frac{\ell}{L}\right)^{2d}. \quad (3.22)$$

Proof. Recalling that $X_\ell(J_L(E)) \in \{0\} \cup \mathbb{N}$, we have

$$\begin{aligned} & \mathbb{P}\{X_\ell(J_L(E)) > m_k\} \\ & \leq \mathbb{P}\{X_\ell(J_L(E)) - m_k \geq 1\} \\ & = \mathbb{P}\{X_\ell(J_L(E))(X_\ell(J_L(E)) - m_k) \geq 1\} \\ & = \mathbb{P}\{\chi_{\{\omega \mid X_\ell(J_L(E)) \geq m_k+1\}} X_\ell(J_L(E))(X_\ell(J_L(E)) - m_k) \geq 1\} \\ & \leq \mathbb{E}\{\chi_{\{\omega \mid X_\ell(J_L(E)) \geq m_k+1\}} X_\ell(J_L(E))(X_\ell(J_L(E)) - m_k) \geq 1\} \\ & \leq C_M \left(\frac{\ell}{L}\right)^{2d}, \end{aligned} \quad (3.23)$$

by the extended Minami estimate [8]. □

3.2. Estimates on the joint probability. We return to considering two scaled intervals $I_L(E)$ and $J_L(E')$, with $E \neq E'$. Because of (3.22), we have

$$\begin{aligned}
& \mathbb{P}\{(X_\ell(I_L(E)) \geq 1) \cap (X_\ell(J_L(E')) \geq 1)\} \\
& \leq \mathbb{P}\{(X_\ell(I_L(E)) \geq m_k + 1) \cap (X_\ell(J_L(E')) \geq m_k + 1)\} \\
& \quad + \mathbb{P}\{(X_\ell(I_L(E)) \leq m_k) \cap (X_\ell(J_L(E')) \geq m_k + 1)\} \\
& \quad + \mathbb{P}\{(X_\ell(I_L(E)) \leq m_k + 1) \cap (X_\ell(J_L(E')) \geq m_k)\} \\
& \quad + \mathbb{P}\{(X_\ell(I_L(E)) \leq m_k) \cap (X_\ell(J_L(E')) \leq m_k)\} \\
& \leq \mathbb{P}\{(X_\ell(I_L(E)) \leq m_k) \cap (X_\ell(J_L(E')) \leq m_k)\} \\
& \quad + C_0 \left(\frac{\ell}{L}\right)^{2d}.
\end{aligned} \tag{3.24}$$

The probability on the last line of (3.24) may be bounded above by

$$\begin{aligned}
& \mathbb{P}\{(X_\ell(I_L(E)) \leq m_k) \cap (X_\ell(J_L(E')) \leq m_k)\} \\
& \leq \sum_{k_1, k_2}^{m_k} \mathbb{P}\{(X_\ell(I_L(E)) = k_1) \cap (X_\ell(J_L(E')) = k_2)\}.
\end{aligned} \tag{3.25}$$

Since m_k is independent of L , it suffices to estimate

$$\mathbb{P}\{(X_\ell(I_L(E)) = k_1) \cap (X_\ell(J_L(E')) = k_2)\}. \tag{3.26}$$

The proof of the next key Proposition 3.1 follows the ideas in [7].

Proposition 3.1. *For $k_1, k_2 = 1, \dots, m_k$ and positive numbers (α, β) satisfying (3.32), we have*

$$\mathbb{P}\{(X_\ell(I_L(E)) = k_1) \cap (X_\ell(J_L(E')) = k_2)\} \leq C \left(\frac{K}{\Delta E - 4d}\right)^4 L^{-2d(1-2\beta-\alpha)}. \tag{3.27}$$

Proof. 1. We begin with some observation concerning the eigenvalue averages. We let $\Omega_0(\ell, k_1, k_2)$ denote the event

$$\Omega_0(\ell, k_1, k_2) := \{\omega \mid (X_\ell(I_L(E)) = k_1) \cap (X_\ell(J_L(E')) = k_2)\} \cap \Omega_{W, \ell}, \tag{3.28}$$

for $k_1, k_2 = 1, \dots, m_k$. The set $\Omega_{W, \ell}$ is the set of ω for which the eigenvalue spacing for $H_{\omega, \ell}$ in the interval $I_L(E)$ or $J_L(E')$ is $\mathcal{O}(\ell^{-d})$. By the Wegner estimate, the probability of this set is at least $1 - C_W(\ell/L)^{-d}$, as discussed in Lemma 2.1. We define the subset $\Delta \subset \Lambda_\ell \times \Lambda_\ell$ by $\Delta := \{(i, i) \mid i \in \Lambda_\ell\}$. For each pair of sites $(i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta$, the Jacobian determinant of the mapping $\varphi : (\omega_i, \omega_j) \rightarrow (\mathcal{T}_\ell(E, k_1), \mathcal{T}_\ell(E', k_2))$, given by:

$$J_{ij}(\mathcal{T}_\ell(E, k_1), \mathcal{T}_\ell(E', k_2)) := \begin{vmatrix} \partial_{\omega_i} \mathcal{T}_\ell(E, k_1) & \partial_{\omega_j} \mathcal{T}_\ell(E, k_1) \\ \partial_{\omega_i} \mathcal{T}_\ell(E', k_2) & \partial_{\omega_j} \mathcal{T}_\ell(E', k_2) \end{vmatrix}. \tag{3.29}$$

As we will show in section 3.3, the condition $J_{ij}(\mathcal{T}_\ell(E, k_1), \mathcal{T}_\ell(E', k_2)) \geq \lambda(L) > 0$ implies that the average of the eigenvalues in $I_L(E)$ and $J_L(E')$ effectively vary independently with respect to any pair of independent random variables (ω_i, ω_j) , for $i \neq j$. We define the following events for pairs $(i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta$:

$$\Omega_0^{i,j}(\ell, k_1, k_2) := \Omega_0(\ell, k_1, k_2) \cap \{\omega \mid J_{ij}(\mathcal{T}_\ell(E, k_1), \mathcal{T}_\ell(E', k_2)) \geq \lambda(L)\}, \tag{3.30}$$

where $\lambda(L) > 0$ is given by

$$\lambda(L) := (\Delta E - 4d)K^{-1}L^{-\beta d}, \quad (3.31)$$

where the exponent $\beta > 0$ satisfies

$$0 < \beta < \frac{1}{2}, \quad 0 < \alpha + 4\beta < 1, \quad 0 < \alpha < \frac{2}{5}\beta. \quad (3.32)$$

For example, we may take $\beta = \frac{1}{8}$ and $\alpha < \frac{1}{20}$.

2. We next compute $\mathbb{P}\{\Omega_0^{i,j}(\ell, k_1, k_2)\}$. Following Klopp [7, pg. 242], we prove in section 3.3 that the positivity of the Jacobian determinant insures that the map φ , restricted to a certain domain, is a diffeomorphism. In particular, for any pair $(i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta$, if $(\omega_i^0, \omega_j^0, \omega_{ij}^\perp) \in \Omega_0^{i,j}(\ell, k_1, k_2)$, then it follows from Lemma 3.2 if $\|(\omega_i^0, \omega_j^0) - (\omega_i, \omega_j)\| > L^{-d}\lambda^{-2}(L)$, consequently one has $(\mathcal{T}_\ell(E, k_1, \omega), \mathcal{T}_\ell(E', k_2, \omega)) \notin I_L(E) \times J_L(E')$. This would contradict the fact that $\omega \in \Omega_0(\ell, k_1, k_2)$. This is key in the following computation:

$$\begin{aligned} & \mathbb{P}\{\Omega_0^{i,j}(\ell, k_1, k_2)\} \\ &= \mathbb{E}_{\omega_{ij}^\perp} \left\{ \int_{\mathbb{R}^2} \chi_{\Omega_0^{i,j}(\ell, k_1, k_2)}(\omega_i, \omega_j, \omega_{ij}^\perp) g(\omega_i) g(\omega_j) d\omega_i d\omega_j \right\} \\ &\leq \mathbb{E}_{\omega_{ij}^\perp} \left\{ \int_{\mathbb{R}^2} \chi_{\{\|(\omega_i, \omega_j) - (\omega_i^0, \omega_j^0)\|_\infty \leq L^{-d}\lambda^{-2}\}}(\omega_i, \omega_j, \omega_{ij}^\perp) g(\omega_i) g(\omega_j) d\omega_i d\omega_j \right\} \\ &\leq CL^{-2d}\lambda^{-4}(L). \end{aligned} \quad (3.33)$$

3. We next bound $\mathbb{P}\{\Omega_0(\ell, k_1, k_2)\}$ in terms of $\mathbb{P}\{\Omega_0^{i,j}(\ell, k_1, k_2)\}$ using [7, Lemma 2.5]. This lemma states that for $(u, v) \in (\mathbb{R}^+)^{2n}$ normalized so that $\|u\|_1 = \|v\|_1 = 1$, we have

$$\max_{j \neq k} \left| \begin{array}{cc} u_j & u_k \\ v_j & v_k \end{array} \right|^2 \geq \frac{1}{4n^5} \|u - v\|_1^2. \quad (3.34)$$

Applying this with $n = (2\ell + 1)^d$, and $u = \nabla_\omega \mathcal{T}(\omega)$ and $v = \nabla_\omega \mathcal{T}'(\omega)$, and recalling the positivity (2.11) in point (3) and the normalization in point (4) of section 2.1, we obtain from (3.34) and (2.15):

$$\begin{aligned} \max_{i \neq j \in \Lambda_\ell} J_{ij}(\mathcal{T}_\ell(E), \mathcal{T}_\ell(E'))^2 &\geq \left(\frac{2^3}{\ell^{5d}} \right) \|\nabla_\omega(\mathcal{T}_\ell(E) - \mathcal{T}_\ell(E'))\|_1^2 \\ &\geq \left(\frac{\Delta E - 4d}{K} \right)^2 \left(\frac{2^3}{\ell^{5d}} \right). \end{aligned} \quad (3.35)$$

We partition the probability space as $\{\omega \mid J_{ij} \geq \lambda(L) \text{ some } (i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta\} \cup \{\omega \mid J_{ij} < \lambda(L) \forall (i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta\}$, where we write J_{ij} for the Jacobian $J_{ij}(\mathcal{T}_\ell(E), \mathcal{T}_\ell(E'))$. Suppose that the second event $\{\omega \mid J_{ij} < \lambda(L) \forall (i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta\}$ occurs, so that from (3.35), we have:

$$\begin{aligned} \lambda(L)^2 &= \left(\frac{C_0}{L^{\beta d}} \right)^2 \geq \max_{i \neq j \in \Lambda_\ell} J_{ij}(\mathcal{T}_\ell(E), \mathcal{T}_\ell(E'))^2 \\ &\geq \left(\frac{2^3}{\ell^{5d}} \right) \|\nabla_\omega(\mathcal{T}_\ell(E) - \mathcal{T}_\ell(E'))\|_1^2. \end{aligned} \quad (3.36)$$

This implies that

$$\|\nabla_\omega(\mathcal{T}_\ell(E) - \mathcal{T}_\ell(E'))\|_1 \leq C_1 L^{-d(\beta-5\alpha/2)}. \quad (3.37)$$

So, provided $0 < \alpha < \frac{2}{5}\beta$, we find that the bound (3.37) implies that the $\nabla_\omega \mathcal{T}_\ell(E)$ is almost collinear with $\nabla_\omega \mathcal{T}_\ell(E')$. This contradicts the lower bound (2.15) as long as $\Delta E > 0$. Consequently, the probability of the second event is zero.

4. It follows from this and the partition of the probability space that

$$\begin{aligned} \mathbb{P}\{\Omega_0(\ell, k_1, k_2)\} &\leq \sum_{(i,j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta} \mathbb{P}\{\Omega_0^{i,j}(\ell, k_1, k_2)\} \\ &\leq \ell^{2d} \lambda^{-4}(L) L^{-2d}. \end{aligned} \quad (3.38)$$

We now take $\ell = L^\alpha$ and $\lambda(L) := (\Delta E - 4d)K^{-1}L^{-\beta d}$, with (α, β) satisfying (3.32). With these choices, and the fact that m_k is independent of L , we obtain the probability

$$\mathbb{P}\{\Omega_0(\ell, k_1, k_2)\} \leq C \left(\frac{K}{\Delta E - 4d} \right)^4 L^{-2d(1-2\beta-\alpha)}. \quad (3.39)$$

For choices α and β with $0 < \alpha + 2\beta < 1$, this shows that

$$\mathbb{P}\{\Omega_0(\ell, k_1, k_2)\} \text{ and } \mathbb{P}\{(X_\ell(I_L(E)) = k_1) \cap (X_\ell(J_L(E')) = k_2)\} \rightarrow 0, \text{ as } L \rightarrow 0,$$

for any integers $k_1, k_2 = 1, \dots, m_k$. This proves, up to the proof of the diffeomorphism property of φ , the main result (1.3). \square

3.3. Proof of the diffeomorphism property. We prove the following lemma on the perturbation of a set of good configurations (ω_i^0, ω_j^0) . Let $\Omega_0(\ell, k_1, k_2)$, $k_1, k_2 = 1, \dots, m_k$ be the set of configurations described in (3.28). Similarly, for any pair of sites $(i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta$, the Jacobian determinant $J_{ij}(\mathcal{T}_\ell(E, k_1), \mathcal{T}_\ell(E', k_2))$ is defined in equation (3.29). We also defined events $\Omega_0^{i,j}(\ell, k_1, k_2)$, for pairs $(i, j) \in \Lambda_\ell \times \Lambda_\ell \setminus \Delta$, in (3.30):

$$\Omega_0^{i,j}(\ell, k_1, k_2) := \Omega_0(\ell, k_1, k_2) \cap \{\omega \mid J_{ij}(\mathcal{T}_\ell(E, k_1), \mathcal{T}_\ell(E', k_2)) \geq \lambda(L)\}, \quad (3.40)$$

where $\lambda(L) > 0$ has the value

$$\lambda(L) := \frac{\Delta E - 4d}{K} L^{-d\beta}, \quad (3.41)$$

where α and β satisfy the constraints in (3.32). The stability estimate following from the diffeomorphism property of φ is given in the following lemma.

Lemma 3.2. [7, Lemma 2.6] *Suppose that $(\omega_i^0, \omega_j^0, \omega_{ij}^\perp) \in \Omega_0^{i,j}(\ell, k_1, k_2)$ and (α, β) satisfy (3.32). Then for any pair $(\omega_i, \omega_j) \in \mathbb{R}^2$ with*

$$\|(\omega_i^0, \omega_j^0) - (\omega_i, \omega_j)\| > L^{-d} \lambda^{-2}(L),$$

one has

$$(\mathcal{T}_\ell(E, k_1, \omega), \mathcal{T}_\ell(E', k_2, \omega)) \notin I_L(E) \times J_L(E'), \quad (3.42)$$

where $\lambda(L)$ has the value given in (3.41).

Proof. 1. Let us fix ω_{ij}^\perp so that $(\omega_i^0, \omega_j^0, \omega_{ij}^\perp) \in \Omega_0^{i,j}(\ell, k_1, k_2)$. We consider the square \mathcal{S} in two-dimensional configuration space:

$$\mathcal{S} := \{(\omega_i, \omega_j) \mid \|(\omega_i^0, \omega_j^0) - (\omega_i, \omega_j)\| \leq L^{-d} \lambda(L)^{-2}\} \quad (3.43)$$

and the map $\varphi : \mathcal{S} \rightarrow \mathbb{R}^2$ defined by

$$\varphi(\omega_i, \omega_j) = (\mathcal{T}_\ell(E, k_1, \omega), \mathcal{T}_\ell(E', k_2, \omega)).$$

The first goal is to prove that φ is an invertible map between \mathcal{S} and its range $\varphi(\mathcal{S})$.

2. To prove that φ is injective, we suppose (ω_i, ω_j) and (ω'_i, ω'_j) both belong to \mathcal{S} and that $\varphi(\omega_i, \omega_j) = \varphi(\omega'_i, \omega'_j)$. Let $D_{ij}\varphi$ denote the 2×2 matrix that is the derivative of φ with respect to (ω_i, ω_j) . By the Fundamental Theorem of Calculus, the definition of \mathcal{S} , and the Hessian estimate (2.20), we have

$$\begin{aligned} & \|D_{ij}\varphi(\omega_i, \omega_j) - D_{ij}\varphi(\omega'_i, \omega'_j)\| \\ & \leq (\|\text{Hess}\mathcal{T}(\omega)\| + \|\text{Hess}\mathcal{T}'(\omega)\|) L^{-d} \lambda(L)^{-2} \leq C_0 L^{-(4-3\alpha-4\beta)d}. \end{aligned} \quad (3.44)$$

The exponent is positive if $0 < \frac{3}{4}\alpha + \beta < 1$ that is satisfied due to (3.32). By a Taylor's expansion and the Hessian estimate (2.20), we obtain

$$\|\varphi(\omega_i, \omega_j) - \varphi(\omega'_i, \omega'_j) - D_{ij}\varphi(\omega_i^0, \omega_j^0) \cdot (\omega - \omega')\| \leq CL^{(3\alpha-2)d} \|(\omega' - \omega)\|^2. \quad (3.45)$$

As a consequence, we can bound the difference

$$\|\varphi(\omega_i, \omega_j) - \varphi(\omega'_i, \omega'_j)\|$$

from below. Recall that the Jacobian determinant of $D_{ij}\varphi(\omega_i^0, \omega_j^0)$ is bounded below by $\lambda(L)$ since $(\omega_i^0, \omega_j^0) \in \mathcal{S}$. For any pair $(\omega_i, \omega_j), (\omega'_i, \omega'_j) \in \mathcal{S}$, we have $\|(\omega_i, \omega_j) - (\omega'_i, \omega'_j)\| \leq CL^{-(1-2\beta)d}$. These facts, the Hessian estimate in (3.44), and the Taylor expansion in (3.45) yield

$$\begin{aligned} \|\varphi(\omega_i, \omega_j) - \varphi(\omega'_i, \omega'_j)\| & \geq |D_{ij}\varphi(\omega_i^0, \omega_j^0) \cdot (\omega' - \omega)| - CL^{(3\alpha-2)d} \|(\omega' - \omega)\|^2 \\ & \geq CL^{-d\beta} \|(\omega' - \omega)\| - CL^{-d(3-3\alpha-2\beta)} \|(\omega' - \omega)\| \\ & \geq C_0(L) \|(\omega' - \omega)\|, \end{aligned} \quad (3.46)$$

where $C_0(L) := C(L^{-d\beta} - L^{-d(3-3\alpha-2\beta)}) > 0$ is strictly positive for $0 < \alpha + \beta < 1$.

1. This proves the injectivity of φ .

3. We next show that φ is an analytic diffeomorphism from \mathcal{S} onto its range. Estimate (3.44) implies that the Jacobians are close:

$$|\text{Jac}\varphi(\omega'_i, \omega'_j) - \text{Jac}\varphi(\omega_i^0, \omega_j^0)| \leq CL^{-(3-3\alpha-2\beta)d}. \quad (3.47)$$

Since $(\omega_i^0, \omega_j^0) \in \Omega_0^{i,j}(\ell, k_1, k_2)$, we know that $|\text{J}_{ij}(\mathcal{T}(\omega^0), \mathcal{T}'(\omega^0))| \geq \lambda(L)$. This lower bound and (3.47) imply that for all $(\omega_i, \omega_j) \in \mathcal{S}$ we have

$$\text{J}_{ij}(\mathcal{T}(\omega), \mathcal{T}'(\omega)) \geq C[L^{-d\beta} - L^{-(3-3\alpha-2\beta)d}] > 0, \quad (3.48)$$

provided $0 < \alpha + \beta < 1$. Consequently, for all $(\omega_i, \omega_j) \in \mathcal{S}$ and L large enough, the Inverse Function Theorem implies that φ is an analytic diffeomorphism. Furthermore, the Jacobian of φ^{-1} satisfies the bound

$$|\text{Jac}\varphi^{-1}(\omega_i, \omega_j)| \leq CL^{d\beta}. \quad (3.49)$$

4. To complete the proof of the lemma, we recall that the map $\omega \rightarrow \mathcal{T}_\ell(E, k_1, \omega)$ is nondecreasing as shown in section 2.1. Hence, we can consider $\|(\omega_i, \omega_j) - (\omega_i^0, \omega_j^0)\|_\infty = L^{-d}\lambda^{-2}(L)$. Let us suppose, to the contrary, that for some such pair $(\omega_i, \omega_j) \in \mathbb{R}^2$ with

$$\|(\omega_i^0, \omega_j^0) - (\omega_i, \omega_j)\| = L^{-d}\lambda^{-2}(L) = CL^{-(1-2\beta)d},$$

one has

$$(\mathcal{T}_\ell(E, k_1, \omega), \mathcal{T}_\ell(E', k_2, \omega)) \in I_L(E) \times J_L(E'). \quad (3.50)$$

Then, using the bound (3.49), we have

$$\begin{aligned} L^{-d}\lambda^{-2}(L) = CL^{-d(1-2\beta)} &< \|(\omega_i^0, \omega_j^0) - (\omega_i, \omega_j)\| \\ &= \|\varphi^{-1}(\mathcal{T}_\ell(E, k_1, \omega), \mathcal{T}_\ell(E', k_2, \omega)) - \varphi^{-1}(E, E')\| \\ &\leq CL^{-d}L^{\beta d} = CL^{-d(1-\beta)}. \end{aligned} \quad (3.51)$$

As $L \rightarrow \infty$, we obtain a contradiction since $\beta > 0$. \square

4. ASYMPTOTICALLY INDEPENDENT RANDOM VARIABLES: PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1. To prove that $\xi_E^\omega(I)$ and $\xi_{E'}^\omega(J)$ are independent, we recall that the limit points ξ_E^ω are the same as those obtained from a certain uniformly asymptotically negligible array ([8, Proposition 4.4]). To obtain this array, we construct a cover of Λ_L by non-overlapping cubes of side length 2ℓ centered at points n_p . We use $\ell = L^\alpha$, where (α, β) satisfy (3.32). For example, we can take $0 < \alpha < 1/20$. The number of such cubes $\Lambda_\ell(n_p)$ is $N_L := [(2L+1)/(2\ell+1)]^d$. The local Hamiltonian is $H_{p,\ell}^\omega$. The associated eigenvalue point process is denoted by $\eta_{\ell,p}^\omega$. We define the point process $\zeta_{\Lambda_L}^\omega = \sum_{p=1}^{N_L} \eta_{p,\ell}^\omega$. For a bounded interval $I \subset \mathbb{R}$, we define the local random variable $\eta_{\ell,p}^\omega(I) := \text{Tr}(E_{H_{p,\ell}^\omega}(I_L(E)))$ and similarly for the scaled interval $J_L(E')$. For $p \neq p'$, these random variables are independent. We compute

$$\begin{aligned} \mathbb{P}\{(\zeta_{\Lambda_L}^\omega(I) \geq 1) \cap (\zeta_{\Lambda_L}^\omega(J) \geq 1)\} &= \sum_{p,p'=1}^{N_L} \mathbb{P}\{(\eta_{\ell,p}^\omega(I) \geq 1) \cap (\eta_{\ell,p'}^\omega(J) \geq 1)\} \\ &= \sum_{p,p'=1}^{N_L} \mathbb{P}\{\eta_{\ell,p}^\omega(I) \geq 1\} \mathbb{P}\{\eta_{\ell,p'}^\omega(J) \geq 1\} \\ &\quad + \mathcal{E}_L(E, E', I, J), \end{aligned} \quad (4.52)$$

where the error term is just the diagonal $p = p'$ contribution:

$$\begin{aligned} \mathcal{E}_L(E, E', I, J) &= \sum_{p=1}^{N_L} [\mathbb{P}\{(\eta_{\ell,p}^\omega(I) \geq 1) \cap (\eta_{\ell,p}^\omega(J) \geq 1)\} \\ &\quad - \mathbb{P}\{\eta_{\ell,p}^\omega(I) \geq 1\} \mathbb{P}\{\eta_{\ell,p}^\omega(J) \geq 1\}]. \end{aligned} \quad (4.53)$$

The first probability on the right side of (4.53) is bounded above by $C_0 L^{-2d(1-2\beta-\alpha)}$ due to the decorrelation estimate (1.3). The bound on the

second probability on the right of (4.53) is $C_W^2 L^{-2d(1-\alpha)}$. It is obtained from the square of the Wegner estimate

$$\mathbb{P}\{\eta_{\ell,E'}^{(p)}(J) \geq 1\} \leq C_W(\ell/L)^d = C_W L^{-d(1-\alpha)}.$$

Since $N_L \sim (L/\ell)^d = L^{(1-\alpha)d}$, we find that

$$\mathcal{E}_L(E, E', I, J) \leq C_W^2 L^{-d(1-\alpha)} + C_0 L^{-d(1-\alpha-4\beta)} \rightarrow 0, \quad L \rightarrow \infty, \quad (4.54)$$

because of (3.32). Since the set of limit points ζ^ω and ξ^ω are the same [8], this estimate proves that

$$\lim_{L \rightarrow \infty} \mathbb{P}\{(\zeta_{E,\Lambda_L}^\omega(I) \geq 1) \cap (\zeta_{E',\Lambda_L}^\omega(J) \geq 1)\} = \mathbb{P}\{\xi_E^\omega(I) \geq 1\} \mathbb{P}\{\xi_{E'}^\omega(J) \geq 1\}, \quad (4.55)$$

establishing the asymptotic independence of the random variables $\xi_E^\omega(I)$ and $\xi_{E'}^\omega(J)$ provided $|E - E'| > 4d$.

5. BOUNDS ON EIGENVALUE MULTIPLICITY

The extended Minami estimate may be used with the Klein-Molchanov argument [6] to bound the multiplicity of eigenvalues in the localization regime. The basic argument of Klein-Molchanov is the following. If H_ω has at least $m_k + 1$ linearly independent eigenfunctions with eigenvalue E in the localization regime, so that the eigenfunctions exhibit rapid decay, then any finite volume operator $H_{\omega,L}$ must have at least $m_k + 1$ eigenvalues close to E for large L . But, by the extended Minami estimate, this event occurs with small probability. The first lemma is a deterministic result based on perturbation theory.

Lemma 5.1. *Suppose that $E \in \sigma(H)$ is an eigenvalue of a self adjoint operator H with multiplicity at least $m_k + 1$. Suppose that all the associated eigenfunctions decay faster than $\langle x \rangle^{-\sigma}$, for some $\sigma > d/2 > 0$. We define $\epsilon_L := CL^{-\sigma + \frac{d}{2}}$. Then for all $L \gg 0$, the local Hamiltonian $H_L := \chi_{\Lambda_L} H \chi_{\Lambda_L}$ has at least $m_k + 1$ eigenvalues in the interval $[E - \epsilon_L, E + \epsilon_L]$.*

Proof. 1. Let $\{\varphi_j \mid j = 1, \dots, M\}$ be an orthonormal basis of the eigenspace for H and eigenvalue E . We assume that the eigenvalue multiplicity $M \geq m_k + 1$. We define the local functions $\varphi_{j,L} := \chi_{\Lambda_L} \varphi_j$, for $j = 1, \dots, M$. These local functions satisfy:

$$\begin{aligned} 1 - \epsilon_L &\leq \|\varphi_{j,L}\| \leq 1, \\ |\langle \varphi_{i,L}, \varphi_{j,L} \rangle| &\leq \epsilon_L, \quad i \neq j. \end{aligned} \quad (5.56)$$

It is easy to check that these conditions imply that the family is linearly independent. Let V_L denote the M -dimensional subspace of $\ell^2(\Lambda_L)$ spanned by these functions.

2. As in [6], it is not difficult to prove that the functions $\varphi_{j,L}$ are approximate eigenfunctions for H_L :

$$\|(H_L - E)\varphi_{j,L}\| \leq \epsilon_L \|\varphi_{j,L}\|. \quad (5.57)$$

Furthermore, for any $\psi_L \in V_L$, we have $\|(H_L - E)\psi_L\| \leq 2\epsilon_L \|\psi_L\|$.

3. Let $J_L := [E - 3\epsilon_L, E + 3\epsilon_L]$. We write P_L for the spectral projector $P_L := \chi_{J_L}(H_L)$ and $Q_L := 1 - P_L$ is the complementary projector. For any $\psi \in V_L$, we have $\|Q_L\psi\| \leq (3\epsilon_L)^{-1} \|(H_L - E)Q_L\psi\| \leq (2/3)\|\psi\|$. Since $\|P_L\psi\|^2 = \|\psi\|^2 - \|Q_L\psi\|^2 \geq (5/9)\|\psi\|^2$, it follows that $P_L : V_L \rightarrow \ell^2(\Lambda_L)$ is injective. Consequently, we have

$$\dim \text{Ran } P_L = \text{Tr}(P_L) \geq \dim V_L = M > m_k.$$

Redefining the constant $C > 0$ in the definition of ϵ_L , we find that H has at least $m_k + 1$ eigenvalues in $[E - \epsilon_L, E + \epsilon_L]$. \square

The second lemma is a probabilistic one and the proof uses the extended Minami estimate.

Lemma 5.2. *Let $I \subset \mathbb{R}$ be a bounded interval. For $q > 2d$, and any interval $J \subset I$ with $|J| \leq L^{-q}$, we define the event*

$$\mathcal{E}_{L,I,q} := \{\omega \mid \text{Tr}(\chi_J(H_{\omega,L})) \leq m_k \ \forall J \subset I, |J| \leq L^{-q}\}. \quad (5.58)$$

Then, the probability of this event satisfies

$$\mathbb{P}\{\mathcal{E}_{L,I,q}\} \geq 1 - C_0 L^{2d-q}. \quad (5.59)$$

Proof. We cover the interval I by $2([L^q|I|/2] + 1)$ subintervals of length $2L^{-q}$ so that any subinterval J of length L^{-q} is contained in one of these. We then have

$$\mathbb{P}\{\mathcal{E}_{L,I,q}^c\} \leq (L^q|I| + 2)\mathbb{P}\{\chi_J(H_{\omega,L}) > m_k\}. \quad (5.60)$$

The probability on the right side is estimated from the extended Minami estimate

$$\mathbb{P}\{\chi_J(H_{\omega,L}) > m_k\} \leq C_M(L^{-q}L^d)^2 = C_M L^{2(d-q)}, \quad (5.61)$$

so that

$$\mathbb{P}\{\mathcal{E}_{L,I,q}^c\} \leq C_M(L^q|I| + 2)L^{2(d-q)} = C_M(|I| + 1)L^{2d-q}. \quad (5.62)$$

This establishes (5.59). \square

Theorem 5.1. *Let H^ω be the generalized Anderson Hamiltonian described in section 1 with perturbations P_i having uniform rank m_k . Then the eigenvalues in the localization regime have multiplicity at most m_k with probability one.*

Proof. We consider a length scale $L_k = 2^k$. It follows from (5.59) that the probability of the complementary event $\mathcal{E}_{L_k,I,q}^c$ is summable. By the Borel-Cantelli Theorem, that means for almost every ω there is a $k(q, \omega)$ so that for all $k > k(q, \omega)$ the event $\mathcal{E}_{L_k,I,q}$ occurs with probability one. Let us suppose that H^ω has an eigenvalue with multiplicity at least $m_k + 1$ in an interval I and that the corresponding eigenfunctions decay exponentially. Then, by Lemma 5.1, the local Hamiltonian H_{ω,L_k} has at least $m_k + 1$ eigenvalues in the interval $[E - \epsilon_L, E + \epsilon_L]$ where $\epsilon_L = CL^{-(\beta - \frac{d}{2})}$, for any $\beta > 5d/2$. This contradicts the event $\mathcal{E}_{L_k,I,q}$ which states that there are no more than m_k eigenvalues in any subinterval $J \subset I$ with $|J| \leq L^{-q}$ since we can find $q > 2d$ so that $\beta - \frac{q}{2} > q$. \square

It appears that the simplicity of eigenvalues in the localization regime might be enough to imply a Minami estimate. Further investigations on the simplicity of eigenvalues for Anderson-type models may be found in the article by Naboko, Nichols, and Stolz [11]

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